In preparation for the upcoming term test, we will do an overview of what we've learned so far.

Functions

Problem 1
Determine if the *maps* below are valid *functions*.

1. $f : \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{x}$.
2. $f : \mathbb{R} \to [0, 2\pi], f(x) = \cos\left(\frac{1}{x^2}\right)$.
3. $f : (0,1) \to \mathbb{N}, f(x) = 2^{x_1}3^{x_2}5^{x_3}7^{x_4}\ldots$, where $0.x_1x_2x_3x_4\ldots$ is the decimal representation of x.
4. $f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} 2x+1 & x \geq 0 \\ 1 - x^2 & x \leq 0 \end{cases}$.

Problem 2
Let $f : B \to C$, $g : A \to B$ be surjective.

1. What is the domain and codomain of $f \circ g$?
2. Show that $f \circ g$ is surjective.
3. Suppose, instead of knowing that f and g are both surjective, that we only know $f \circ g$ is surjective. Must f be surjective? Must g be surjective?

Suprema/Infima

Problem 3

1. State the *completeness axiom*.
2. Show that the completeness axiom doesn't hold if $\mathbb{R}$ is replaced with $\mathbb{Q}$.

Problem 4
Let $S \subseteq \mathbb{R}$. Give two equivalent definitions for "$M = \sup S$". Give two equivalent definitions for "$m = \inf S$".

Problem 5
Show that $\sup(-\infty, x) = x$ for any $x \in \mathbb{R}$.

Density of Rationals

Problem 6
Let

$$h : \mathbb{R} \to \mathbb{R}, h(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ x & x \in \mathbb{Q}. \end{cases}$$

Show that $\sup\{h(x) : x \in (a,b)\} = b$, for any open interval $(a,b) \subseteq \mathbb{R}, a < b$, where $b > 0$.

Problem 7

In this question, we provide a proof that the *irrationals* are *dense*.

1. Define what it means for a set $S \subseteq \mathbb{R}$ to be dense.
2. Define *countable* and *uncountable* sets. Recall that $\mathbb{Q}$ is countable, while $\mathbb{R}$ is uncountable.
3. Show that $|\mathbb{R}| = |(-\frac{\pi}{2}, \frac{\pi}{2})|$ by defining a bijection between them. If you prefer, you may draw a graph instead of explicitly defining this bijection. This shows $(-\frac{\pi}{2}, \frac{\pi}{2})$ is uncountable.
4. Show that $|(a, b)| = |(-\frac{\pi}{2}, \frac{\pi}{2})|$ for any $a < b$. This shows any open interval (a, b) is uncountable.
5. Prove by contradiction that any open interval (a, b) contains an irrational number. Conclude that the irrationals are dense.